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A Note on A New Improved Poisson Distribution and Its Approximations

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Abstract

This work, determine a new improved Poisson distribution with mean $\mu = \frac{nr}{b}$ from Polya distribution with parameters r, b, n and c where r, b, n and c are non-negative integer. It was found that the new improved Poisson approximates binomial distribution accurately more than the normal Poisson for n so large and $\frac{r}{r+b}$ sufficient small for $x \in \{0,1 \dots n\}$ while the new improved Poisson approximates Polya distribution for a certain random variables $x \in \{0.1\}$, provided that n is not large enough. This implies the new improved Poisson distribution is not sufficient to enough to approximate Polya distribution.

Keywords- *Binomial distribution, Poisson distribution, A new Improved Poisson*

1.0 INTRODUCTION

Let X be a non- negative integer –valued random variable such that $0 \le x \le n$ is of the form

$$P_Y(x) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{b}{c}+n-x-1}{n-x}}{\binom{\frac{r+b}{c}+n-1}{n}} \quad x = 0,1, \dots \dots n \qquad 1.0$$

The mean and variance is given as $\frac{rn}{r+b}$ and $\frac{nrb(r+b+cn)}{(r+b)^2(r+b+c)}$ respectively

The probability function in (1.0) can be expressed as of the form

$$P_Y(x) = \binom{n}{x}\frac{(r.c)_{x-1}(b,c)_{n-x-1}}{(r+b,c)_{n-1}} \quad x = 0,1 \dots . n \qquad 1.1$$

Let X be a non –negative valued random variable that has a Binomial distribution with parameters n, and $\frac{r}{r+b}$. Its probability function can be expressed as

$$B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n} \quad x = 0,1 \dots n \qquad 1.2$$

By a taking the limit as $r, r+b \to \infty$ as $\frac{r}{r+b}$ remain constant $\mathrm{P}_{Y}(x) \to B\ (x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$ $x = 0,1 \ldots n$

And also if $n \to \infty, \frac{r}{r+b} \to 0$ and $\mu = n\frac{r}{r+b}$ remain constant then $B\ (x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n} \to \wp_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1 \ldots n$

Samson et al [1], [4] gave a new Improved Poisson in extension of Teerapabolarn [8] with mean $\frac{NA}{B}$ by deriving from the Generalized Binomial distribution as of the form

$$G_{bd}(A, B, N) \cong \frac{\wp_\lambda(x)\left(\frac{B}{A+B}\right)^N e^\lambda}{1+\frac{x(x-1)}{2N}} \tag{1.3}$$

Where $\wp_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ Samson et al [1], [4] used (1.3) to generate a model for determing the prices of options.

This work , focus on determing a new improved Poisson from Polya distribution with $\mu = \frac{rn}{b}$ for approximation of Polya distribution and Binomial distribution respectively , in extension of Samson et al [1] ,[4].The level of accuracy is given in the point metric form .

2.0 METHOD

The tools for giving the result are Polya distribution. Polya distribution used in this study was discussed by Teerapabolarn [11].It is a discrete distribution that depends on four parameters N, n, r and c where N, n, r and $c \in$ N and the mean and the variance of X are $\mu = \frac{nr}{r+b}$ and $\sigma^2 = \frac{nrb(r+b+cn)}{(r+b)^2(r+b+C)}$ .The probability distribution of a random variable X taking non- negative value x, such that $0 \leq x \leq n$ is of the form

$$\mathrm{P}_{Y}(x) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{b}{c}+n-x-1}{n-x}}{\binom{\frac{r+b}{c}+n-1}{n}} \quad x = 0,1, \ldots \ldots n \tag{2.0}$$

From (2.0) we obtained

$$\mathrm{P}_{Y}(x) = \binom{n}{x}\frac{[r(r+c)\ldots.r+(x-1)c][b(b+c)\ldots b+(n-x-1)c]}{[(r+b)\ldots.r+b+(n-1)c]}$$

$$= \binom{n}{x}\frac{[r\ldots r+(x-1)c][b\ldots.b+(n-x-1)c]}{[r+b\ldots r+b+(n-1)c]}$$

$$= \binom{n}{x}\frac{\left[\frac{r}{c}\ldots\frac{r}{c}+\beta\right]\left[\frac{b}{c}\ldots\frac{b}{c}+\vartheta\right]}{\frac{r+b}{c}\ldots\frac{r+b}{c}+(n-1)}$$

where $\beta = \begin{cases} 0 \ \ if\ x = 0 \\ (x-1)\ \ if\ x = 1,2..n \end{cases}$ and $\vartheta = \begin{cases} 0\ if\ x = 0,1,..n \\ (n-x-1)\ if\ x = 0 \end{cases}$ for $c \geq 1$

$$P_Y(x) = \binom{n}{x} \frac{r+b\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right] r+b\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{r+b\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

$$= \binom{n}{x} \frac{(r+b)^x\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right](r+b)^{n-x}\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{(r+b)^n\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

$$= \binom{n}{x} \frac{(r+b)^x(r+b)^{n-x}\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right]\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{(r+b)^n\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

$$= \binom{n}{x} \frac{\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right]\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

If $r, r+b \to \infty$, while $\frac{r}{r+b}$ remain constant

$$P_Y(x) \to B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

If X is Binomially distributed then

$$B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

$$= \binom{n}{x}\frac{r^x}{(r+b)^x}\cdot\frac{b^{n-x}}{(r+b)^{n-x}}$$

$$B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\left(\frac{r}{r+b}\right)^x\left(\frac{b}{r+b}\right)^{n-x} \qquad 2.1$$

Where $\lambda = \frac{nr}{r+b}$ and $\frac{r}{r+b} = \frac{\lambda}{n}$ then (2.1) becomes

$$B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{x!}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

As $n \to \infty$

$$B\,(x)_{n,\frac{r}{r+b}} \to \frac{\lambda^x e^{-\lambda}}{x!} = \wp_\lambda(x)$$

This is an indication that Polya and Binomial distribution can be approximated by Poisson distribution under certain conditions on their parameters .

Lemma 2.0 Let $x \in$ N , we have the following Teerapabolarn [10]

$$\prod_{i=0}^{x-1}\left(1-\frac{i}{n}\right)=\frac{1}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)}$$

3.0 RESULT

Theorem : For $x \in \mathrm{N} \cup \{0\}$ and $\mu=\frac{nr}{b}$ for $r, r+b \to \infty$ then

$$\mathrm{P}_{\Upsilon}(x) \cong \widetilde{\wp_{\lambda}(x)} \quad \text{and} \quad B\,(x)_{n,\frac{r}{r+b}} \cong \widetilde{\wp_{\lambda}(x)} \text{ where } \widetilde{\wp_{\lambda}(x)}=\frac{\frac{\lambda^x}{x!}\left(\frac{b}{r+b}\right)^n}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)}$$

Proof

For $x=0$ and $r, r+b \to \infty$

$$\mathrm{P}_{\Upsilon}(x) \cong B\,(x)_{n,\frac{r}{r+b}}=\binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

$$\mathrm{P}_{\Upsilon}(0) \cong B\,(0)_{n,\frac{r}{r+b}}=\binom{n}{0}\frac{r^0 b^{n-x}}{(r+b)^n}=\left(\frac{b}{r+b}\right)^n$$

$$\mathrm{P}_{\Upsilon}(x) \cong \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

$$=\binom{n}{x}\left(\frac{n\frac{r}{r+b}}{n}\right)^x\left(\frac{b}{r+b}\right)^n\left(\frac{b}{r+b}\right)^{-x}=\binom{n}{x}\left(\frac{n\frac{r}{r+b}}{n}\right)^x \times \frac{\left(\frac{b}{r+b}\right)^n}{\left(\frac{b}{r+b}\right)^x}=\binom{n}{x}\frac{n^x\left(\frac{r}{r+b}\right)^x}{n^x}\frac{\left(\frac{b}{r+b}\right)^n}{\frac{\left(\frac{n\frac{r}{r+b}}{n\frac{r}{r+b}}\right)^x}{\frac{b}{r+b}}}$$

$$=\binom{n}{x}\frac{n^x\left(\frac{r}{r+b}\right)^x}{n^x}\frac{\left(\frac{b}{r+b}\right)^n}{\left(\frac{n\frac{r}{r+b}}{\lambda}\right)^x}=\binom{n}{x}\frac{n^x\left(\frac{r}{r+b}\right)^x}{n^x}\left(\frac{b}{r+b}\right)^n \times \frac{\lambda^x}{n^x\left(\frac{r}{r+b}\right)^x}$$

$$=\frac{\lambda^x}{x!}\prod_{i=0}^{x-1}\left(1-\frac{i}{N}\right)\left(\frac{b}{r+b}\right)^n=\frac{\lambda^x}{x!}\frac{1}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n}\right)}\left(\frac{b}{r+b}\right)^n \text{ by lemma 2.0}$$

$$\mathrm{P}_{\Upsilon}(x) \cong \frac{\frac{\lambda^x}{x!}\left(\frac{b}{r+b}\right)^n}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)} \cong B\,(x)_{n,\frac{r}{r+b}}$$

$\mathrm{P}_{\Upsilon}(x) = \widetilde{\wp_{\lambda}(x)}$, but if n is large $O\left(\frac{1}{n}\right) \approx 0$

4. NUMERICAL EXAMPLES

The following numerical examples are given to illustrate how well the new improved Poisson distribution with $\mu = \frac{nr}{b}$ approximates both Binomial and Polya distribution respectively with parameters n and $\frac{r}{r+b}$.

Example 4.1 : Suppose $n = 80, \quad \frac{r}{r+b} = 0.01, r + b = 100, c = 1, \mu = 0.808080808$

x	$P_Y(x)$	$B(x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert P_Y(x) - \widetilde{\wp_\lambda(x)}$	$\left\lvert B(x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)}\right\rvert$	$\left\lvert B(x)_{n,\frac{r}{r+}} - \wp_\lambda(x)\right\rvert$	$\lvert P_Y(x) - \wp_\lambda(x)$
0	No Result	0.44752321	0.44752321	0.44571265	No Result	0.00000000	0.00181056	No Result
1	No Result	0.36163492	0.36163492	0.36017184	No Result	0.00000000	0.00146308	No Result
2	No Result	0.14428868	0.14431123	0.14552398	No Result	0.00002255	0.00123530	No Result
3	No Result	0.03789400	0.03793504	0.03919838	No Result	0.00000410 5	0.00130438	No Result
4	No Result	0.00736828	0.00739631	0.00791886	No Result	0.00002803	0.00055059	No Result
5	No Result	0.00113129	0.00114224	0.00127982	No Result	0.00001094	0.00014853	No Result

Example 4.2 : Suppose $n = 10, r + b = 1000, \frac{r}{r+b} = 0.01 \quad \mu = 0.101010101$, then numerical results are as follows

x	$P_Y(x)$	$B(x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert P_Y(x) - \widetilde{\wp_\lambda(x)}\rvert$	$\left\lvert B(x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)}\right\rvert$	$\left\lvert B(x)_{n,\frac{r}{r+b}} - \wp_\lambda(x)\right\rvert$	$\lvert P_Y(x) - \wp_\lambda(x)\rvert$
0	0.904790682	0.904382075	0.903923902	0.904837418	0.000408607	0.000000000	0.000455343	0.000004674
1	0.090569634	0.091351724	0.091351724	0.090483742	0.000782090	0.000000000	0.000867982	0.000085892
2	0.004492181	0.004152351	0.004194294	0.004524419	0.000297887	0.000041943	0.000371836	0.000032006
3	0.000144182	0.000111847	0.000119496	0.000150806	0.000024686	0.000007648	0.000038958	0.000006624
4	0.000000329	0.000001977	0.000002452	0.000003770	0.000000841	0.000000475	0.000001793	0.000000477

5	0.000000056	0.000000024	0.000000040	0.000000075	0.000000016	0.000000016	0.000000051	0.000000019

Example 4.3 : Suppose $n = 300$, $r + b = 100$, $\frac{r}{r+b} = 0.01$ $\mu = 3.03030303$, then numerical results are as follows

x	$P_Y(x)$	$B(x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert P_Y(x) - \widetilde{\wp_\lambda(x)}$	$\lvert B(x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)}$	$\lvert B(x)_{n,\frac{r}{r+b}} - \wp_\lambda(x)\rvert$	$\lvert P_Y(x) - \wp_\lambda(x)$
0	No Result	0.04904089	0.04904089	0.04830100	No Result	0.00000000	0.00073989	No Result
1	No Result	0.14860877	0.14860877	0.14636666	No Result	0.00000000	0.00224211	No Result
2	No Result	0.22441425	0.22441675	0.22176767	No Result	0.00000249	0.00265658	No Result
3	No Result	0.22516986	0.22518732	0,22400775	No Result	0.00001746	0.00116211	No Result
4	No Result	0.16887739	0.16892394	0.16970284	No Result	0.00004655	0.00082545	No Result
5	No Result	0.10098527	0.10105714	0.10285021	No Result	0.00007187	0.00186494	No Result

Example 4.4: Suppose $n = 10, r = 5, r + b = 100$, $\frac{r}{r+b} = 0.05, \mu = 0.526315789$

x	$P_Y(x)$	$B(x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert P_Y(x) - \widetilde{\wp_\lambda(x)}$	$\lvert B(x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)}$	$\lvert B(x)_{n,\frac{r}{r+b}} - \wp_\lambda(x)\rvert$	$\lvert P_Y(x) - \wp_\lambda(x)\rvert$
0	0.612207502	0.598736939	0.598736939	0.606530660	0.013470563	0.0000000	0.007793721	0.005676842
1	0.294330530	0.315124705	0.315124705	0.303265330	0.020794175	0.0000000	0.011859375	0.008934800
2	0.077154605	0.074634799	0.075388683	0.075816332	0.001765922	0.000753884	0.001181533	0.001338273
3	0.016265635	0.010475089	0.011191303	0.012636055	0.005074332	0.000716214	0.002160966	0.003629580
4	0.001957200	0.000964805	0.001196439	0.001579507	0.000760761	0.000231634	0.000614702	0.000377693
5	0.000211378	0.000060935	0.000100753	0.000157951	0.000110625	0.000039818	0.000053427	0.000053427

5.0 DISCUSSION

From example 4.1-4.3 , it was found that the new improved Poisson is sufficient enough to approximate Binomial more the normal Poisson distribution provided $\frac{r}{r+b}$ is small , but not sufficient to approximate Polya distribution.

6. CONCLUSION

In this work , the new improved Poisson distribution with $\mu = \frac{rr}{b}$ is obtained from Polya distribution with parameters r, b, n and c. The result obtained gives a good approximation to Binomial distribution provided $\frac{r}{r+b}$ is small, but not sufficient enough to give a good approximation to Polya distribution. By comparison new improved Poisson remain the best approximation to Binomial and while normal Poisson remains the best approximation to Polya distribution more the new improved Poisson distribution .

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